J. Fluid Mech. (1969), vol. 35, part 1, pp. 53–75
Printed in Great Britain

A point explosion in a cold exponential atmosphere

By DALLAS D. LAUMBACH AND RONALD F. PROBSTEIN
Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received 2 May 1968)

The problem considered is that of a strong shock propagating from a point energy source into a cold atmosphere whose density varies exponentially with altitude. An explicit analytic solution is obtained by taking the flow field as 'locally radial' and using an integral method with an energy constraint. A scaling law is given which eliminates the parametric dependence of the solution on the explosion energy, scale height, and atmospheric density at the point of the explosion. The scaling law also transforms the infinity of solutions for various polar angles into two distinct solutions which show that all motions of the ascending portion of the shock may be scaled from the vertically upward behaviour and all motions of the descending portion of the shock may be scaled from the vertically downward behaviour. The limit in the lateral direction of both of the fundamental solutions corresponds to the case of the uniform density atmosphere. The results for the uniform density atmosphere show remarkable agreement with the exact Taylor–Sedov results. Comparison with finite difference calculations of Troutman & Davis for the vertically upward and downward directions shows excellent agreement with respect to the prediction of shock propagation velocity, position, and the flow variables behind the shock. A scaling law for the time, shock velocity, and pressure for different values of the adiabatic exponent γ is proposed which correlates the results of the present analysis for different values of γ over the entire range of shock positions where the analysis applies. The solution shows that, contrary to the result obtained by Kompaneets, there is no theoretical limit as to how far downward a strong shock may propagate. The far field behaviour of the shock wave in the upward and downward directions is found to be of the same form as the self-similar asymptotic solutions obtained by Raizer for a plane shock. It is shown by relaxing the energy constraint in the vertically downward direction that the asymptotic result obtained agrees closely with that obtained by Raizer. The energy constraint, however, is the appropriate one for all but the far field behaviour. The far field limit of the present solution in the upward direction is found to compare favourably with the approximate asymptotic calculations of Hayes for an ascending curved shock. The empirical concept of 'modified Sachs scaling' for calculating the overpressure is considered and shown within this model to have a justification in the downward direction but a limited range of applicability in the upward direction.

1. Introduction

The problem considered is that of a strong shock propagating from a point source into an inhomogeneous atmosphere whose density varies exponentially with altitude (see Zel'dovich & Raizer 1966). It was first treated by Karlikov (1955) by linearizing about the corresponding solution obtained by Sedov (1946) and Taylor (1950) for a uniform density atmosphere. Kompaneets (1960) analysed the problem approximately by assuming a uniform pressure throughout the flow field behind the shock wave. This pressure was taken to be proportional to the average energy density in the volume bounded by the shock wave and differed from the pressure immediately behind the shock by a factor of λ. The value of λ was estimated from the Taylor–Sedov uniform atmosphere solution and was dependent only on the constant adiabatic exponent. The uniform 'sea of pressure' was assumed to drive the flow field mass which was taken to be contained in a thin layer behind the shock surface. The solution was improved upon by Andriankin *et al.* (1962), who calculated a local λ, which was a function of both time and polar angle. Raizer (1963, 1964) obtained self-similar solutions for a plane rising and descending shock in an exponential atmosphere which he used to describe the asymptotic far field behaviour of the upper and lower parts of the shock wave originating from a point source. Apparently unaware of the previous work of Raizer, Grover & Hardy (1966) published the self-similar solution for a plane shock wave travelling upward along with some numerical calculations. Hayes (1968*a*) determined the self-similar asymptotic solutions in a medium with exponentially varying ray-tube area as well as density. Hayes (1968*b*) attempted to determine the local effect of curvature on Raizer's asymptotic rising shock solution by means of an approximate calculation based on the use of the Chester, Chisnell, Whitham shock propagation law, but with the exponents taken from the results of Hayes (1968*a*).

The purposes of the present paper are: (*a*) to obtain an approximate analytic solution valid for all times and to deduce scaling laws based on the solution; (*b*) to compare the results obtained with exact numerical calculations; (*c*) to see if the Kompaneets model is justified and to see if there is, as predicted by Kompaneets, a limit as to how far downward a strong shock can propagate; (*d*) to compare the far field results of the present analysis with the asymptotic limits of Raizer and Hayes; (*e*) to investigate the empirical concept of 'modified Sachs scaling' for calculating the overpressure.

2. Theory

In the present analysis the shock wave is assumed sufficiently strong that counterpressure may be neglected and the strong shock relations can be applied. The gas is considered to be a calorically and thermally perfect one characterized by an adiabatic exponent γ. Body forces due to the earth's gravitational and magnetic fields, wind effects, and heat transfer by radiation and conduction are neglected. The atmosphere is thus considered to be initially at rest and cold, i.e. at zero temperature and pressure, with a density distribution given by the exponential law

$$\rho_0 = \rho_B \exp(-h/\Delta). \tag{1}$$

Here, Δ is the scale height of the atmosphere taken to be a constant, ρ_0 the atmospheric density, ρ_B the density at the burst point, and h the altitude measured positive upward from the point of explosion.

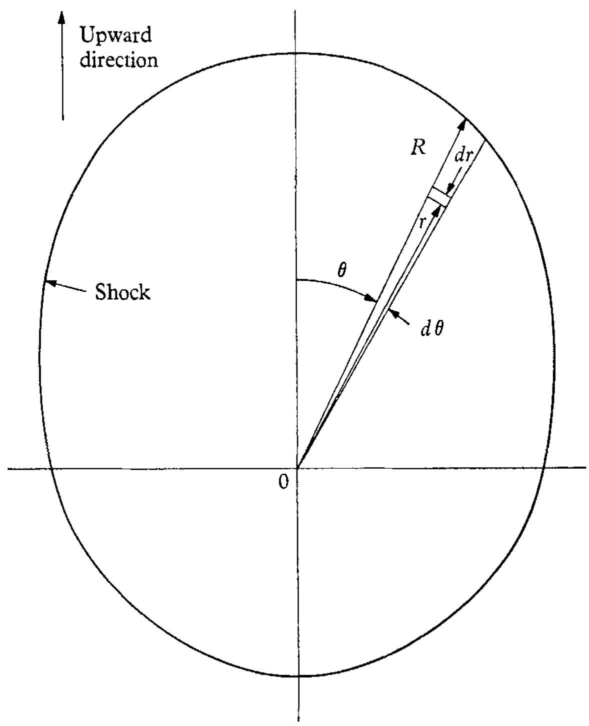

FIGURE 1. Flow geometry.

In figure 1 a sketch is shown of the shock envelope at a given time after explosion along with the polar co-ordinate system used in the present analysis. The basic assumption to be employed in the theory is that the flow field is 'locally radial'. By this is meant that we may neglect gradients in the θ direction, where θ is the polar angle measured from the vertical axis as shown in figure 1. Such an assumption corresponds to considering the streamlines from the origin as straight. It should be noted that for large shock radii (of about 4–5 scale heights) as the shock becomes increasingly asymmetric, the assumption of negligible gradients in the θ direction will be less satisfactory. We would also note, however, that the strong shock assumption becomes invalid by the time the shock has propagated about 4–6 scale heights, except for $\theta < \frac{1}{4}\pi$, even for the largest energy sources of practical interest (see, e.g. Troutman & Davis 1965).

Under the preceding assumptions, the problem is axisymmetric about the vertical axis through the energy release point, designated as the origin in figure 1. Here r is the Eulerian co-ordinate of a fluid particle of thickness dr, and $R(t; \theta)$ is the position of the shock front at a given polar angle θ. In the present problem it was found convenient to utilize a Lagrangian formulation. The exponential density distribution given by (1) can be written in terms of the Lagrangian co-ordinate r_0 as

$$\rho_0 = \rho_B \exp[-(r_0/\Delta)\cos\theta], \tag{2}$$

with r_0 defined to be the position of a particular fluid particle at the burst time, $t = 0$.

With the assumption of local radiality the continuity equation for any polar angle is simply

$$\rho_0 r_0^2 dr_0 = \rho r^2 dr. \tag{3}$$

The radial momentum equation when expressed in Lagrangian co-ordinates by means of (3) is

$$\frac{\partial^2 r}{\partial t^2} + \frac{r^2}{\rho_0 r_0^2}\frac{\partial p}{\partial r_0} = 0. \tag{4}$$

Finally, for a perfect gas with the assumption that the entropy of a fluid particle remains constant after passage through the shock, the energy equation reduces to

$$\frac{p(r_0,t)}{p_s(r_0)} = \left[\frac{\rho(r_0,t)}{\rho_s(r_0)}\right]^\gamma, \tag{5}$$

where the subscript s refers to conditions when the particle is at the shock front. With the strong shock assumption we have

$$\rho_s = \frac{\gamma+1}{\gamma-1}\rho_0 \tag{6}$$

and

$$p_s = \frac{2}{\gamma+1}\rho_0 \dot{R}^2, \tag{7}$$

with the dot over the shock location R denoting differentiation with respect to time.

The independent variables in the problem are the Lagrangian co-ordinate r_0 and the time t. The problem considered is not a self-similar one since it contains the characteristic length scale Δ, and there is a fixed point in space from which it can be measured (the energy-release point). An exact analytic solution therefore appears difficult to obtain and we instead seek an approximate solution by an integral method of the type discussed by Hayes & Probstein (1966).

We begin by writing the momentum equation (4) in integral form

$$p(r_0,t;\theta) - p_s(R;\theta) = \int_{r_0}^{R} \frac{1}{r^2}\frac{\partial^2 r}{\partial t^2}\rho_0 \bar{r}_0^2 d\bar{r}_0, \tag{8}$$

where p_s is the pressure immediately behind the shock front and $\bar{r}_0$ is a dummy variable. Consistent with the assumption that the mass contained within a differential solid angle is constant, the integral energy conservation equation (see e.g. Hayes & Probstein 1966) for a given polar angle may be expressed through (3) by

$$\frac{E}{4\pi} = \int_0^R \frac{p}{\gamma-1} r^2 dr + \int_0^R \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^2 dr_0, \tag{9}$$

where we have for later convenience written the first integral in Eulerian form. Here E is the total hydrodynamic energy of the flow field, considered to be known and constant, since heat transfer by radiation out through the shock front and counterpressure have been neglected. In (9) the first term on the right-hand side represents the internal energy per unit solid angle and the second term the kinetic energy per unit solid angle.

We next consider how, based on the nature of the flow, the integrands in (8) and (9) may be approximated. Chernyi (1957, 1959, 1960, 1961) has suggested their determination by a perturbation expansion about their values at the shock front using the strong shock-density ratio $\epsilon = \rho_0/\rho_s$ given by (6) as the small parameter. Thus, in one approach, the Eulerian co-ordinate r is expanded as a power series in ϵ in which the coefficients are functions of the time t alone. The justification for this method is that for a strong shock most of the flow is concentrated in a thin layer next to the shock, while the remaining region of the flow field is essentially at constant pressure. This technique has been found to work remarkably well in a number of problems as, for example, in the point-explosion problem in a uniform atmosphere. We would note that in the present problem when the shock is propagating downward ($\frac{1}{2}\pi < \theta \leqslant \pi$) into a region of increasing density the effect of the mass being concentrated at the shock will be intensified. On the other hand, when the shock is propagating upward ($0 \leqslant \theta < \frac{1}{2}\pi$) the effect will be diminished and it cannot be assumed in advance that the mass will always be concentrated at the shock front. Of course, whether this is the case can be checked at intermediate stages of the solution.

In the present analysis we shall make use of the idea that most of the mass is concentrated near the shock. However, we shall not employ the method of Chernyi since it has the limitation that even though the expansion is about the shock front, terms involving second and higher derivatives of r with respect to t are not exact even there, except in the limit $\epsilon \to 0$. Instead we shall use a scheme whereby these quantities are evaluated exactly at the shock. To do this we expand the Eulerian co-ordinate r in a Taylor series about the shock front, so that the integrals in (8) and (9) can be approximated in a manner analogous to that of Stokes' method of stationary phase and Laplace's method. We that note r is a function of the independent variables, r_0 and t. However, rather than expanding directly in terms of these variables we consider a Taylor expansion in the Lagrangian co-ordinate r_0, with the expansion parameterized in the time t through the Taylor coefficients and R. Thus we have for $r(r_0, t)$

$$r(r_0,t) = R + \left.\frac{\partial r}{\partial r_0}\right|_R (r_0 - R) + \frac{1}{2}\left.\frac{\partial^2 r}{\partial r_0^2}\right|_R (r_0 - R)^2 + \dots. \tag{10}$$

In the present analysis we shall retain terms only to the order of $(r_0 - R)^2$. By differentiating (10), satisfying the conservation equations and boundary conditions, and assuming that most of the mass is concentrated near the shock front so that $(r - R) \ll R$ for $r_0 \sim R$, we then obtain the Eulerian co-ordinate, the velocity, and the acceleration immediately behind the shock front expressed in terms of R and its derivatives with respect to time.

From the continuity equation (3) and strong shock condition (6) we have that

$$\left.\frac{\partial r}{\partial r_0}\right|_R = \left.\frac{\rho_0 r_0^2}{\rho r^2}\right|_R = \frac{\gamma - 1}{\gamma + 1}. \tag{11}$$

Using the above relation and differentiating (10) with respect to t we have

$$r = R+\left(\frac{\gamma-1}{\gamma+1}\right)(r_0-R)+\frac{1}{2}\frac{\partial^2 r}{\partial r_0^2}\bigg|_R (r_0-R)^2, \tag{12a}$$

$$\frac{\partial r}{\partial t} = \frac{2}{\gamma+1}\dot{R}-\frac{\partial^2 r}{\partial r_0^2}\bigg|_R (r_0-R)\dot{R}, \tag{12b}$$

$$\frac{\partial^2 r}{\partial t^2} = \frac{2}{\gamma+1}\ddot{R}+\frac{\partial^2 r}{\partial r_0^2}\bigg|_R \dot{R}^2-\frac{\partial^2 r}{\partial r_0^2}\bigg|_R (r_0-R)\ddot{R}. \tag{12c}$$

The first two of the above relations yield the obvious results for the Eulerian co-ordinate and velocity behind the shock front, while the acceleration is found from (12c) and (2)–(7)

$$r_s = R, \tag{13a}$$

$$\left(\frac{\partial r}{\partial t}\right)_s = \frac{2}{\gamma+1}\dot{R}, \tag{13b}$$

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = \frac{4(2\gamma-1)}{(\gamma+1)^2}\ddot{R}-\frac{2(\gamma-1)}{(\gamma+1)^2}\frac{\cos\theta}{\Delta}\dot{R}^2+\frac{8\gamma(\gamma-1)}{(\gamma+1)^3}\frac{\dot{R}^2}{R}. \tag{13c}$$

The evaluation of (13c), which is parametrized in θ, is somewhat lengthy and is given in the appendix.

We now approximate $r^{-2}(\partial^2 r/\partial t^2)$ in the integrand of (8) by the values at the shock given by (13), replace ρ_0 by (2), and on carrying out the integration obtain

$$p-p_s = \left(\frac{2}{\gamma+1}\right)^2\left(\frac{\Delta}{\cos\theta}\right)^2\frac{\rho_B}{\eta^3}\left\{2(2\gamma-1)\eta\ddot{\eta}-(\gamma-1)\eta\dot{\eta}^2+4\gamma\left(\frac{\gamma-1}{\gamma+1}\right)\dot{\eta}^2\right\}$$
$$\times\left\{\exp\left(-\frac{r_0}{R}\eta\right)\left[\left(\frac{r_0}{R}\right)^2\frac{\eta^2}{2}+\left(\frac{r_0}{R}\right)\eta+1\right]-e^{-\eta}\left[\frac{\eta^2}{2}+\eta+1\right]\right\}. \tag{14}$$

Here we have defined the basic reduced variable

$$\eta = (R/\Delta)\cos\theta, \tag{15}$$

with the pressure behind the shock p_s given from (2) and (7) by

$$p_s = \frac{2\rho_B}{\gamma+1}\left(\frac{\Delta}{\cos\theta}\right)^2\dot{\eta}^2 e^{-\eta}. \tag{16}$$

To evaluate the energy integral (9) we again make use of the fact that to the order of the present approximation r is not a function of r_0, so that for any r different from R, the corresponding value of r_0 is zero. This is equivalent to the statement that all of the mass is pulled forward behind the shock such that the only remaining mass for $r < R$ is the differential mass originally in the vicinity of $r_0 = 0$. Thus $p(r,t) = p(0,t)$ and the internal energy term of (9) is readily evaluated by substituting the pressure relation given by (14) with r_0 set equal to zero. The kinetic-energy term in (9) is evaluated by replacing ρ_0 through (2), and $(\partial r/\partial t)$ by its value at the shock (13b). After carrying out the indicated integrations and combining terms the following ordinary second-order differential equation is obtained for η as a function of time:

$$f(\eta)\ddot{\eta}+g(\eta)\dot{\eta}^2 = \frac{E}{4\pi\rho_B}\left(\frac{\cos\theta}{\Delta}\right)^5, \tag{17}$$

where $f(\eta)$ and $g(\eta)$ are given by

$$f(\eta) = \frac{8}{3}\frac{(2\gamma-1)\eta}{(\gamma-1)(\gamma+1)^2}[1-e^{-\eta}(\tfrac{1}{2}\eta^2+\eta+1)], \tag{18a}$$

$$g(\eta) = \frac{2}{3}\frac{\eta^3 e^{-\eta}}{(\gamma-1)(\gamma+1)}+\frac{\gamma-1}{2(2\gamma-1)}\left[\frac{7\gamma+3}{(\gamma+1)\eta}-1\right]f(\eta). \tag{18b}$$

It is clear that the parametric dependence upon θ in (17) can be almost eliminated by reducing the time scale, and that upon E, Δ and ρ_B entirely eliminated, by means of the transformation

$$t^* = t\left[\frac{E|\cos^5\theta|}{4\pi\rho_B\Delta^5}\right]^{\frac{1}{2}}. \tag{19}$$

Equation (17) then becomes

$$f(\eta)\eta''+g(\eta)\eta'^2 = 1 \quad \text{for} \quad 0 \leqslant \theta \leqslant \pi/2, \tag{20a}$$

$$f(\eta)\eta''+g(\eta)\eta'^2 = -1 \quad \text{for} \quad \pi/2 \leqslant \theta \leqslant \pi, \tag{20b}$$

where the prime is used here and in what follows to denote differentiation with respect to the reduced time t^*. It can be seen that the introduction of the length scale $\Delta/\cos\theta$ and the time scale $(4\pi\rho_B\Delta^5/E|\cos^5\theta|)^{\frac{1}{2}}$ enables all motions of the ascending portion of the shock ($\theta < \frac{1}{2}\pi$) to be scaled from a single solution of (20*a*) for this régime (say, e.g. $\theta = 0$) and all motions of the descending portion of the shock ($\theta > \frac{1}{2}\pi$) to be scaled from a single solution of (20*b*) for this régime (say, e.g. $\theta = \pi$). Either solution can, of course, also be scaled to arbitrary values of E, Δ and ρ_B.

Since equations (20) are autonomous we can take η as the independent variable and reduce them to the linear first-order equations in η'^2

$$\frac{f(\eta)}{2}\frac{d\eta'^2}{d\eta}+g(\eta)\eta'^2 = \pm 1. \tag{21}$$

The solutions of (21) are given by

$$\eta'^2 = \pm 2\exp\left\{-2\int_a^\eta \frac{g(z)}{f(z)}dz\right\}\int_0^\eta \exp\left\{2\int_a^y \frac{g(z)}{f(z)}dz\right\}\frac{dy}{f(y)}, \tag{22}$$

where y and z are variables of integration, a is a zero of the indicated integrals and $\gamma > 1$. The upper and lower signs in (21) and (22) refer to the upward and downward directions, respectively. We have here set the constants of integration equal to zero to satisfy the requirement that the solutions reduce to the uniform density solution as $\eta \to 0$. The determination of $\eta = \eta(t^*)$ follows from (22) and the relation

$$t^* = \int_0^\eta \frac{d\bar{\eta}}{\bar{\eta}'}, \tag{23}$$

with $\bar{\eta}$ a dummy variable.

With η and η' known, the pressure behind the shock can be determined as a function of r_0 at a particular time from (14) and (16). The density as a function of the Lagrangian co-ordinate can then be determined from (5). By integrating (3) the relation between r and r_0 is given by

$$\frac{r}{R} = \left[3\int_0^{r_0/R} \left(\frac{\rho_0}{\rho}\right)\left(\frac{\bar{r}_0}{R}\right)^2 d\left(\frac{\bar{r}_0}{R}\right)\right]^{\frac{1}{3}}. \tag{24}$$

From these results, the pressure and the density behind the shock front are found as a function of the Eulerian co-ordinate r.

3. Comparison with other solutions

We first compare our results with the exact solution of Taylor (1950) and Sedov (1946) for the case of a uniform density atmosphere. Within the present model, this solution is obtained by considering the limit $\eta = R\cos\theta/\Delta \to 0$. In this limit, the behaviour of (17) is given in dimensional variables by

$$\ddot{R} + \frac{\gamma(5\gamma+1)}{(2\gamma-1)(\gamma+1)}\frac{\dot{R}^2}{R} = \frac{9(\gamma-1)(\gamma+1)^2}{16(2\gamma-1)}\frac{E}{\pi\rho_B}\frac{1}{R^4}. \tag{25}$$

This equation is readily integrated by the same procedure used to obtain (22). The result for the shock velocity as a function of shock location is

$$\dot{R}^2 = \frac{9}{8}\frac{E}{\pi\rho_B}\frac{(\gamma-1)(\gamma+1)^3}{(4\gamma^2-\gamma+3)}\frac{1}{R^3}, \tag{26}$$

which when integrated gives the shock location

$$R = \left[\frac{225}{32}\frac{E}{\pi\rho_B}\frac{(\gamma-1)(\gamma+1)^3}{(4\gamma^2-\gamma+3)}\right]^{\frac{1}{5}} t^{\frac{2}{5}}. \tag{27}$$

Equation (27) differs from the exact Taylor–Sedov results (see, e.g. Sedov 1957) by 1·8 % for $\gamma = 1{\cdot}2$ and 2·3 % for $\gamma = 1{\cdot}4$. The approximate results of Chernyi (1957, 1960) differ by 0·9 % and 1·4 %, respectively. A comparison of the predictions for the pressure behind the shock front for $\gamma = 1{\cdot}4$ is shown in figure 2. In this figure, the pressure is reduced by the shock pressure p_s and the Eulerian co-ordinate by the shock position R. The present analysis gives a pressure ratio at the origin $p(0,t)/p_s$ of 0·379, which compares with the exact Taylor–Sedov value of 0·366, and the approximate Chernyi value of 0·400.

Turning now to the results for the exponential atmosphere, the shock velocity $\dot{R}$ is given as a function of shock location R for $\gamma = 1{\cdot}1$, 1·2, and 1·4 in figure 3 for the ascending portion of the shock and in figure 4 for the descending portion. Both $\dot{R}$ and R have been reduced following the scalings of (15) and (19). Also shown are the results of finite difference calculations reported by Troutman & Davis (1965) for the vertically ascending and descending parts of the shock with $\gamma = 1{\cdot}4$. The Troutman–Davis calculations, which are for a specific yield, scale height, and atmospheric density at the burst point, have been appropriately non-dimensionalized. The calculations were carried out only up to the time when the upper part of the shock had travelled three scale heights from the origin. Up

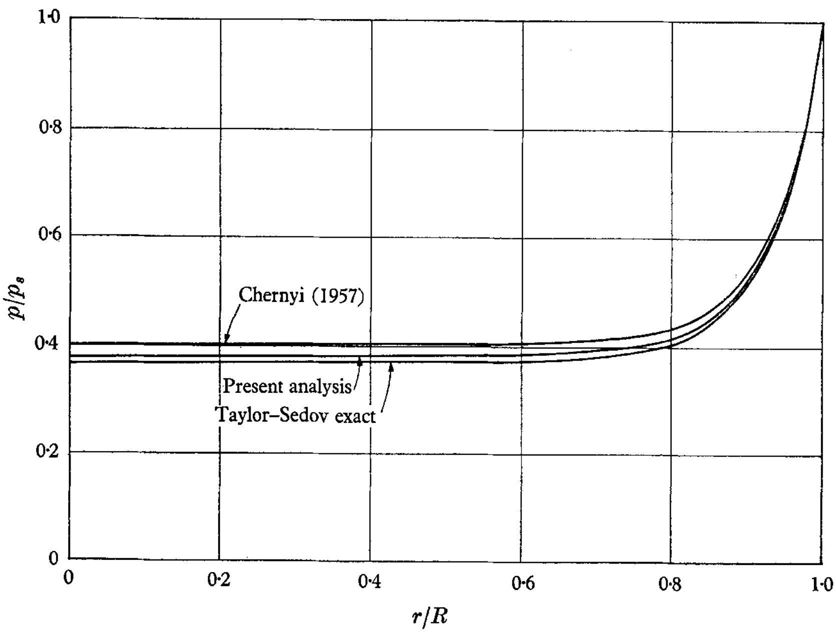

FIGURE 2. Pressure distribution in Eulerian co-ordinates for uniform density atmosphere and $\gamma = 1{\cdot}4$.

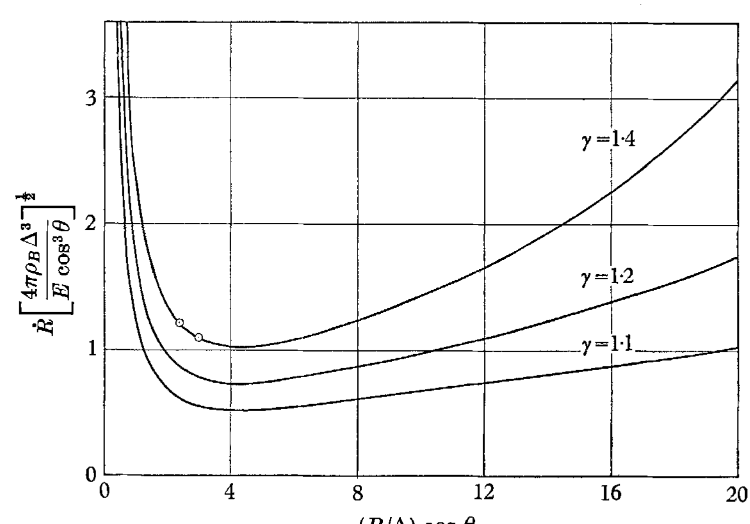

FIGURE 3. Shock velocity in upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$) as a function of shock position. $\odot$, Troutman & Davis, numerical ($\theta = 0$).

to this time the agreement with the results of the present analysis is excellent. It should be pointed out that the Troutman–Davis calculations (SAP code), which were made only for $\theta = 0$ and π, essentially embodied the locally radial approximation for these two locations and were thus one-dimensional. However, these authors compared their results with complete two-dimensional calculations (ROC code for γ constant and variable, and Shell Oil code for γ variable) and report little difference over the range of their calculations.

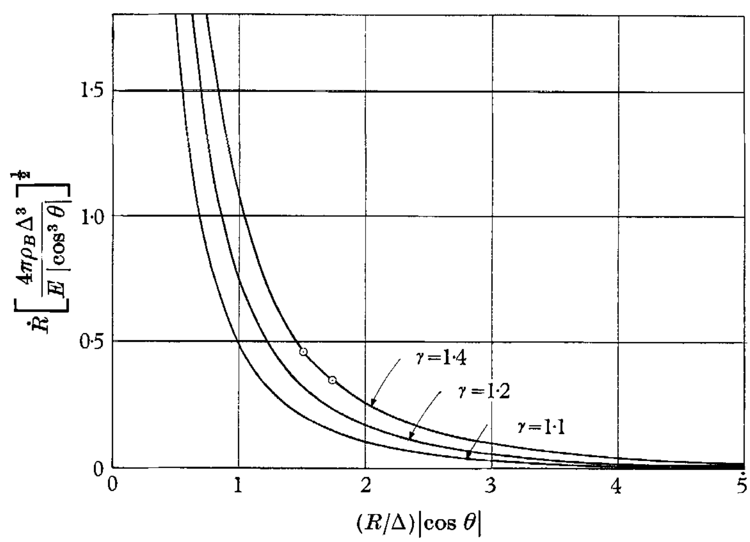

FIGURE 4. Shock velocity in downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$) as a function of shock position. ⊙ Troutman & Davis, numerical ($\theta = \pi$).

It can be seen from figure 3 that for the ascending shock the shock velocity does in fact reach some minimum value beyond which it accelerates to infinity as was predicted in the calculations of Kompaneets (1960) and Andriankin *et al.* (1962). Thus the wave first decelerates as it moves away from the point of the explosion until the atmospheric density decrease ahead of the shock begins to affect the motion appreciably. As seen from figure 4, the descending shock does not experience any theoretical limit beyond which it cannot remain strong, provided, of course, that there is a sufficiently large energy source. The existence of such a limit was predicted by Kompaneets and Andriankin *et al.* Such a limit is a direct consequence of the assumption that a uniform 'sea of pressure' extends throughout the flow field, which drives a thin layer of mass present on the boundaries of the shock envelope. Such an assumption permits no finite pressure gradients in the momentum equation, and hence when the shock propagating upward accelerates off to infinity in a finite time and 'blowout' occurs, the immediate result is that there is no longer any pressure to drive the shock wave propagating in the downward direction. This is simply an inherent limitation of the model. In solutions, such as the present one, the force driving the shock wave downward is not reduced to zero upon the emergence of the shock wave at infinity due to the existence of a finite pressure gradient behind the descending shock. As will be shown later, however, the pressure gradient behind the ascending shock does go to zero at blowout within the present model.

From figures 3 and 4 it can be seen that decreasing γ reduces the shock velocity and this is easily understood from the energy equation (9). Examination of (9) shows that as γ approaches 1 an increasing fraction of the total hydrodynamic

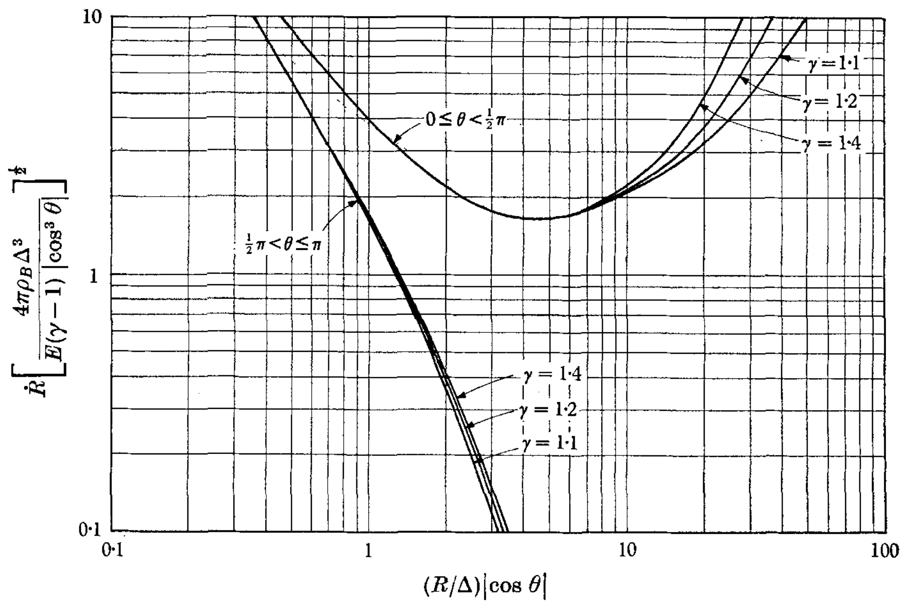

FIGURE 5. Correlation with γ of shock velocity as a function of shock position.

energy is exhibited as internal energy, thereby reducing the kinetic energy of the flow field.

Equation (25) for the uniform atmosphere limit shows that the primary effect of the parameter γ on the shock behaviour in this case could be scaled out if the time scale were to be reduced by $(\gamma-1)^{\frac{1}{2}}$. Since the shock behaves initially as one in a uniform atmosphere this suggests that the principal effect of variations in γ, at least for not too large times, could be eliminated through the introduction of the reduced time

$$t^{**} = t^{*}(\gamma-1)^{\frac{1}{2}}, \tag{28}$$

where t^* is given by (19). When the density is not uniform it can be seen from (18) that by this transformation the $\gamma-1$ factor can be scaled out of the coefficient $f(\eta)$ and the first but not the second term of $g(\eta)$. As will be shown below, however, the coefficient $g(\eta)$ is in fact inversely proportional to $(\gamma-1)$ in the downward direction for η large but not in the upward direction. Therefore the scaling proposed should work well in the downward direction for all times but should be less satisfactory in the upward direction for the far field solution.

In figure 5 we have replotted the results of figures 3 and 4 but with the shock velocity reduced by the additional factor of $(\gamma-1)^{\frac{1}{2}}$. What is apparent is that the results for the different values of γ are very well correlated for all times in the downward direction and up to quite large times in the upward direction ($t^{**} \sim 12$;

this value is obtained from the results of figure 12 to be given below). This correlation extends well beyond the range of times (or scale heights) where the model has its greatest applicability. The correlation is seen to bear out the features of the scaling deduced from the basic equations.

The consequence of the approximate scaling exhibited by (28) is that at a given η for a fixed ρ_B, Δ and θ we have the simple scaling relations

$$\dot{\eta} \propto (\gamma-1)^{\frac{1}{2}}E^{\frac{1}{2}}, \tag{29a}$$

$$p_s \propto (\gamma-1)E. \tag{29b}$$

We have here included the explosion energy E in the scaling since it shows clearly that the effect of decreasing γ is to lower the effective yield.

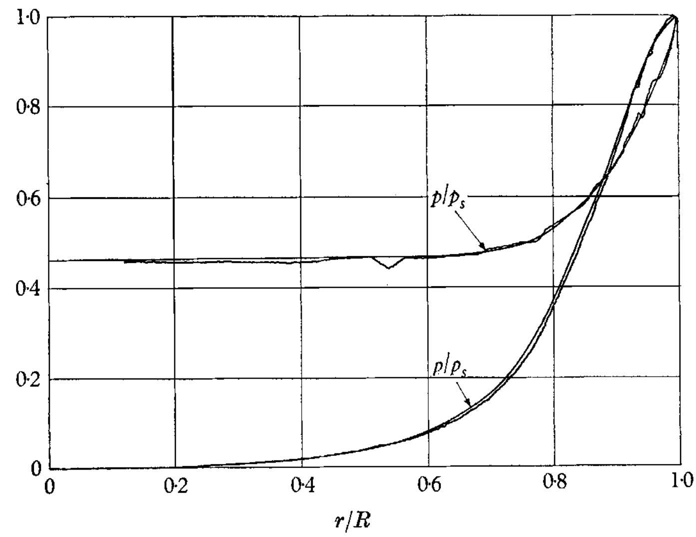

FIGURE 6. Pressure and density distribution in Eulerian co-ordinates at $R\cos\theta/\Delta = 3{\cdot}0$ for $\gamma = 1{\cdot}4$; upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$). ——, present analysis; 〰, Troutman & Davis, numerical ($\theta = 0$).

The flow field pressure and density behind the ascending shock are shown as a function of r/R in figure 6 for $R\cos\theta/\Delta = 3{\cdot}0$ and $\gamma = 1{\cdot}4$. The results are compared with the finite difference calculations of Troutman & Davis (1965), appropriately non-dimensionalized, and, as with the shock velocity, the agreement is exceptionally good. The flow field pressure and density for the descending shock as a function of r/R are shown in figure 7 for $R\cos\theta/\Delta = -1{\cdot}5$. The Troutman–Davis calculations are plotted for comparison and it is seen that there is an even higher degree of agreement in this direction.

To see the effect of the exponential density distribution on the shape of the shock envelope, the shock position for $\gamma = 1{\cdot}2$ has been plotted at four different dimensionless times T in figure 8. The length scale is here measured in scale heights Δ, and the time scale in units of $(4\pi\rho_B\Delta^5/E)^{\frac{1}{2}}$. Comparison is made with one of the shock envelope curves determined by Andriankin *et al.* (denoted by *AKKK*) and it is seen to differ from the present analysis by a factor of almost 2 at $\theta = 0$.

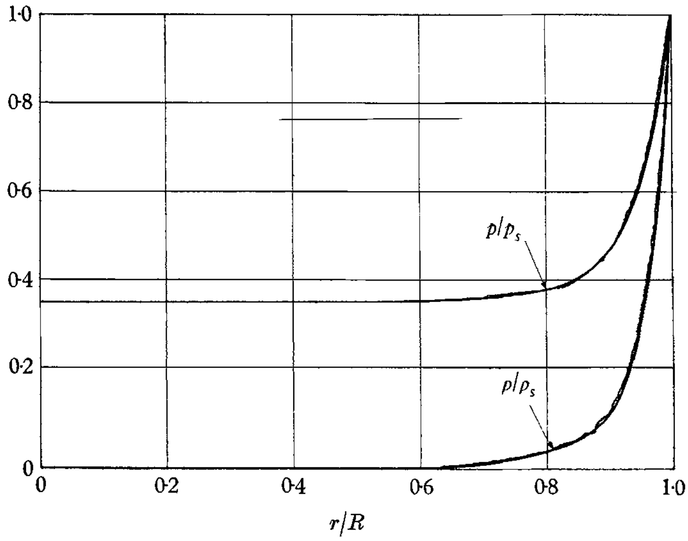

FIGURE 7. Pressure and density distribution in Eulerian coordinates at $R\cos\theta/\Delta = -1{\cdot}5$ for $\gamma = 1{\cdot}4$; downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$). ——, present analysis; ∼∼, Troutman & Davis, numerical ($\theta = \pi$).

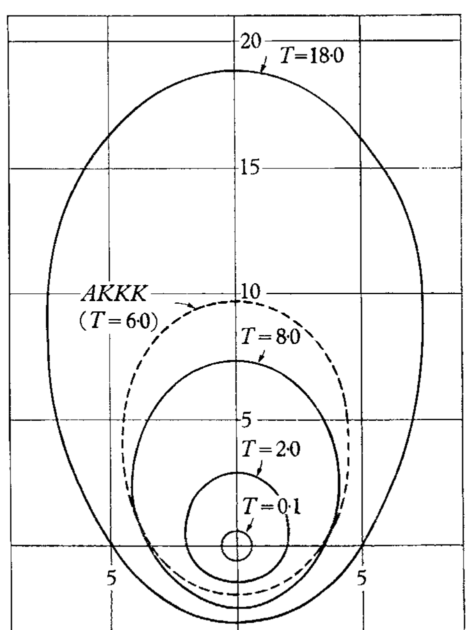

FIGURE 8. Shock envelope at different times for $\gamma = 1{\cdot}2$, with length scale Δ and $T = t(E/4\pi\rho_B\Delta^5)^{\frac{1}{2}}$.

An indication of the regions in which the local radiality assumption breaks down can be obtained from this plot by considering the angle between the tangent to the shock envelope and the radius vector. When this angle differs appreciably from $\frac{1}{2}\pi$, the assumption has broken down, since the θ gradients will no longer be negligible. At $T = 18{\cdot}0$ it appears that the assumption is invalid in the region $\frac{1}{6}\pi < \theta < \frac{2}{3}\pi$, but at $T = 8{\cdot}0$ it still appears to be valid for all θ.

We next consider the asymptotic or far field behaviour of the shock in the upward and downward directions. It should be borne in mind, however, that any 'exact' far field results should not necessarily agree with those of the present analysis, inasmuch as the assumptions invoked in the model become less satisfactory at large times.

In the downward direction as $-\eta$ becomes large, the asymptotic form of (17) is given by

$$\ddot{\eta} - \frac{\gamma}{2\gamma-1}\dot{\eta}^2 = \frac{3E(\gamma-1)(\gamma+1)^2}{16\pi\rho_B(2\gamma-1)}\left(\frac{\cos\theta}{\Delta}\right)^5\frac{e^\eta}{\eta^3}. \tag{30}$$

Writing (30) as a first-order equation in $\dot{\eta}^2$ and integrating,

$$\dot{\eta}^2 \sim -\frac{3E(\gamma-1)(\gamma+1)^2}{8\pi\rho_B}\left(\frac{\cos\theta}{\Delta}\right)^5\frac{e^\eta}{\eta^3}\left[1+O\left(\frac{1}{\eta}\right)\right]+k_1\exp\left(\frac{2\gamma}{2\gamma-1}\eta\right), \tag{31}$$

where k_1 is a constant of integration. Since $\gamma > 1$, the term in k_1 is asymptotically small with respect to the energy-dependent term so that the asymptotic solution becomes

$$\dot{\eta}^2 \sim -\frac{3E(\gamma-1)(\gamma+1)^2}{8\pi\rho_B}\left(\frac{\cos\theta}{\Delta}\right)^5\frac{e^\eta}{\eta^3}\left[1+O\left(\frac{1}{\eta}\right)\right]. \tag{32}$$

A second integration yields the dominant term for η

$$\eta = (R/\Delta)\cos\theta \sim -2\ln t. \tag{33}$$

In the notation of Raizer and Hayes we have for the vertically downward motion

$$\dot{R}|\cos\theta| \sim \alpha\Delta/t = 2\Delta/t. \tag{34}$$

Thus under the constant energy constraint of the present model we find $\alpha = 2$ for all $\gamma > 1$. This result agrees with the asymptotic self-similar solution obtained by Hayes (1968*a*) for a plane shock travelling downward subject to a conservation of energy condition. The fact that α is here found to be independent of γ should be considered a characteristic of the model. We should note that the far field asymptotic behaviour of the descending shock should approach that of a plane shock since under the assumption of straight streamlines (local radiality) the cross-sectional area of the flow increases as R^2. The far field results of the present analysis are therefore appropriately compared with exact asymptotic plane shock solutions. The question of the constraints to be imposed on such solutions must, however, be dictated by the physics of the problem in question.

The descending shock velocity as a function of time, as obtained from combining the results of the indicated integrations in (22) and (23), is plotted in figure 9 for $\gamma = 1{\cdot}4$. Shown for comparison is the self-similar, asymptotic plane shock result obtained by Raizer (1963). The relevant boundary condition on this solution is that the pressure be zero at an infinite distance behind the shock where the velocity goes to minus infinity and in addition, in order for the solution to be single valued, that it pass through a saddle point. This latter condition corresponds physically to having the back pressure behind the shock low enough in order to satisfy the saddle-point criterion. The result of Raizer is seen to lie below the asymptote to the present solution, in which we have required that the total energy per unit solid angle remains constant. From physical considerations it is clear that this is the appropriate constraint for the near field behaviour, say up to 2–3 scale heights vertically downward. However, for the far-field behaviour

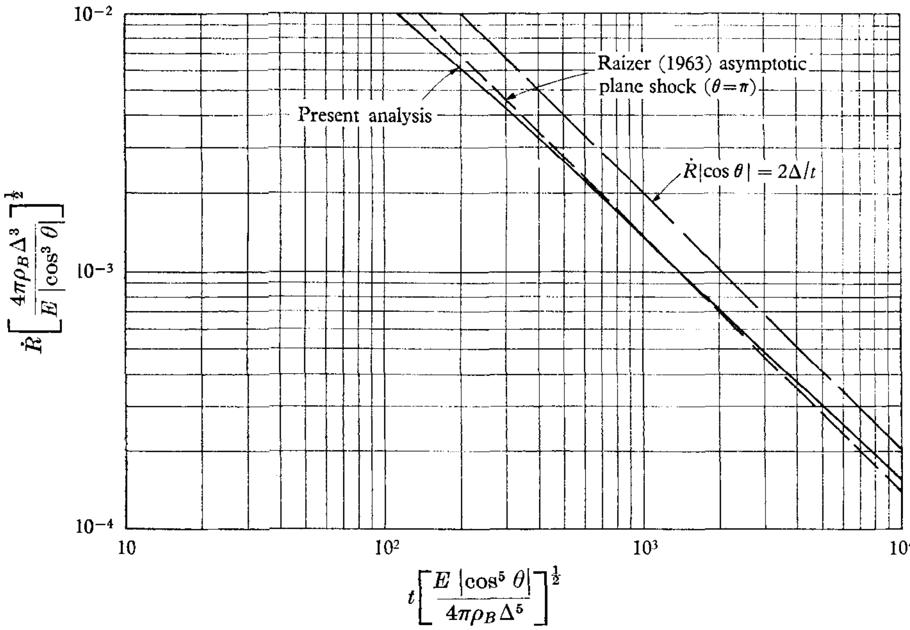

FIGURE 9. Shock velocity in downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$) as a function of time for $\gamma = 1{\cdot}4$.

the constraint will break down since the energy in a given radial slice will no longer remain constant. This is evidenced by the fact that the pressure at $r_0 = 0$ approaches $p_s/(\gamma+1)$ as $\eta \to -\infty$, a result which follows from (14), (16) and (32).

It is interesting to note that if the energy constraint is relaxed such that the energy becomes arbitrary (and therefore can be set equal to zero) the asymptotic solution for $\bar{\eta}^2$ is given by the second term in (31) and we have

$$\alpha = (2\gamma - 1)/\gamma. \tag{35}$$

From (14) and (16) this second solution corresponds to the condition that $p = 0$ at $r_0 = 0$. This condition is then very nearly analogous to the constraint imposed by Raizer (1963). The values of α^{-1} given by (35) are compared in figure 10 with those corresponding to Raizer's solution as calculated by Hayes (1968*a*). The comparison is seen to be quite good with exact agreement at $\gamma = 1{\cdot}0$ and $2{\cdot}0$. In the limiting case $\gamma \to 1$ we might expect the agreement to be exact since in that case the back pressure in Raizer's solution will tend to zero. We would emphasize again, however, that the energy constraint which was used is the physically meaningful one over the range of times where the present analysis may be expected to apply.

In the upward direction as η becomes large, the asymptotic form of (17) is given by

$$\ddot{\eta}-\frac{\gamma-1}{2(2\gamma-1)}\dot{\eta}^2=\frac{3E(\gamma-1)(\gamma+1)^2}{32\pi\rho_B(2\gamma-1)\eta}\left(\frac{\cos\theta}{\Delta}\right)^5. \tag{36}$$

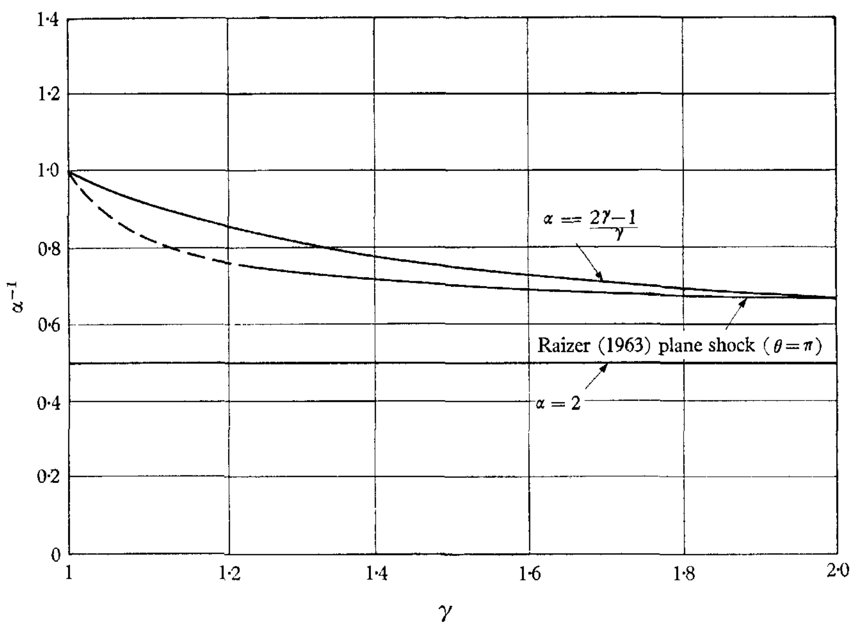

FIGURE 10. Dependence of α on γ in downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$). – – –, extrapolated.

Integrating as before we have asymptotically

$$\dot{\eta}^2 \sim -\frac{3E}{16\pi\rho_B}\frac{(\gamma+1)^2}{\eta}\left(\frac{\cos\theta}{\Delta}\right)^2\left[1+O\left(\frac{1}{\eta}\right)\right]+k_2\exp\left(\frac{\gamma-1}{2\gamma-1}\eta\right), \tag{37}$$

where k_2 is an integration constant. Thus for the upward direction the energy dependent term is asymptotically small compared with the second term on the right-hand side. Hence the asymptotic solution for $\dot{\eta}^2$ becomes

$$\dot{\eta}^2 \sim k_2\exp\left(\frac{\gamma-1}{2\gamma-1}\eta\right). \tag{38}$$

A second integration of (38) gives

$$\eta=\frac{R}{\Delta}\cos\theta \sim -\frac{2(2\gamma-1)}{\gamma-1}\ln\left[\frac{(\gamma-1)k_2^{\frac{1}{2}}}{2(2\gamma-1)}(\tau-t)\right], \tag{39}$$

with τ a constant of integration.

In the notation of Raizer and Hayes we have

$$\dot{R}\cos\theta \sim \frac{\alpha\Delta}{-t_{RH}}=\frac{2(2\gamma-1)}{\gamma-1}\frac{\Delta}{\tau-t}. \tag{40}$$

The time τ is seen to be the time after burst when $\dot{R}\cos\theta$ becomes infinite and the shock wave emerges at infinity. In both Raizer's and Hayes' analyses the time t_{RH} takes on only negative values and in terms of the positive time after burst t used in the present analysis it is given by

$$t_{RH} = t - \tau. \tag{41}$$

The present model thus gives for α the value

$$\alpha = \frac{2(2\gamma-1)}{\gamma-1}. \tag{42}$$

From (42) we find for $\gamma = 1{\cdot}4$ a value of $\alpha = 9{\cdot}00$, which is 12 % larger than the value of 7·89 obtained by Hayes (1968*b*) for a curved shock based on an approximate method. The method is one which uses a shock propagation law of the form suggested by the Chester, Chisnell and Whitham approximation, but with the exponents taken from the self-similar exponentially varying ray-tube area solutions of Hayes (1968*a*). The basic assumption in this procedure is that of local similarity. The curved shock results of Hayes which apply to a shock travelling in a ray tube with an exponentially varying area would therefore be expected to differ from the present model where we assume that the ray-tube area increases as R^2. Therefore, as pointed out previously, the far field results of the present analysis are more properly compared with asymptotic plane shock solutions. Hayes (1968*a*) has determined the value of α corresponding to the exact, self-similar plane shock solution of Raizer (1964) to be 5·45 for $\gamma = 1{\cdot}4$, which differs from the value given from (42) by 39 %.

The difference between the asymptotic rising shock solution of the present model and that of Raizer's plane shock solution can best be seen by considering the different physical pictures presented by the solutions. In the asymptotic 'self-propagating' solution of Raizer (1964) the pressure immediately behind the shock is low, while it is high at some critical distance not too far behind the shock where the saddle point is located. In Raizer's solution therefore the shock is driven by the pressure gradient behind the shock. On the other hand, the present asymptotic solution given by (38) and (39) corresponds to a zero pressure gradient behind the shock. This can be seen from (14), where we note that at large η the bracketed terms in $\ddot{\eta}$ and $\dot{\eta}^2$ cancel identically. The asymptotic zero pressure gradient condition corresponds, of course, to a zero particle acceleration and this may be seen from (13*c*), where we note that the right-hand side is, to within a constant factor, the same as the corresponding terms which cancel in (14). Thus, despite the fact that the flow is non-accelerating, the shock wave itself is accelerating with its acceleration determined by a shock velocity which leads to a non-accelerating flow but a self-propagating shock. It follows that within the present model the shock acceleration off to infinity is essentially a local effect and is independent of the energy of the source from which it originates. It is not clear that the constraints analogous to Raizer's of a saddle-point condition and the condition that the pressure be infinite at an infinite distance behind the shock where the velocity is zero could be imposed on the present model.

The ascending shock velocity as a function of time is determined by combining the results of the integrations indicated in (22) and (23) and is plotted in figure 11 for $\gamma = 1{\cdot}4$. The approximate curved shock asymptote of Hayes (1968*b*) is shown along with the exact plane shock asymptote of Raizer (1964) (as calculated by Hayes 1968*a*) and equation (40). The dimensionless blowout time,

$$\tau[E|\cos^5\theta|/4\pi\rho_B\Delta^5]^{\frac{1}{2}}$$

is found to be 16·16. Figure 12 shows the same quantities for $\gamma = 1{\cdot}2$ with the dimensionless blowout time equal to 28·11. Also shown in figure 12 for comparison is the Andriankin *et al.* (1962) solution, which is seen to agree only qualitatively with the other solutions.

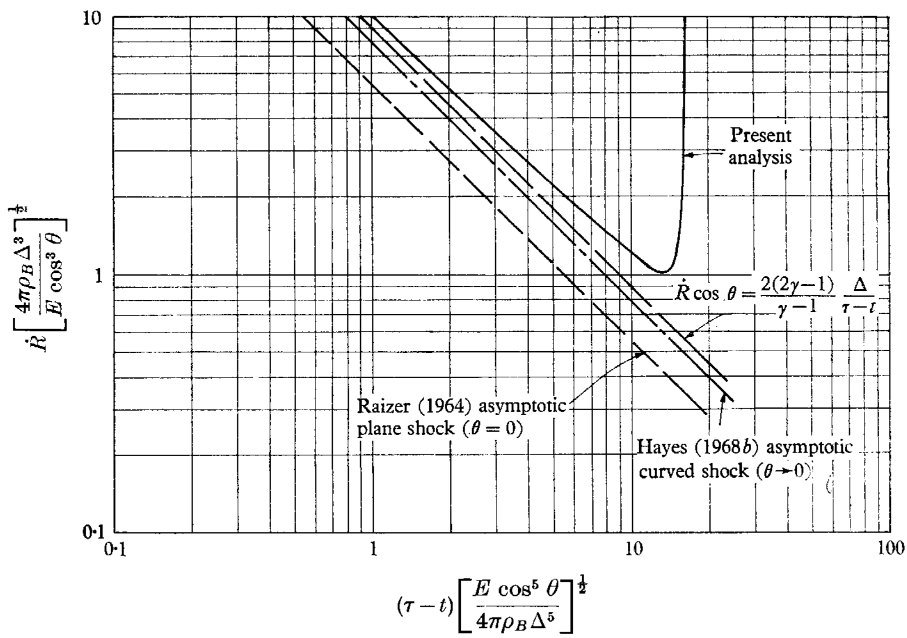

FIGURE 11. Shock velocity in upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$) as a function of time before blowout for $\gamma = 1{\cdot}4$; blowout time $\tau(E\cos^5\theta/4\pi\rho_B\Delta^5)^{\frac{1}{2}} = 16{\cdot}16$.

It can be seen from figures 11 and 12 that the deviation between the values of α predicted by the present analysis and that of Raizer increases for decreasing γ. To show this behaviour more clearly we have in figure 13 plotted α^{-1} as a function of γ and compared our values with the Raizer plane shock values (as calculated by Hayes 1968*a* and Grover & Hardy 1966). The curved-shock asymptotic results of Hayes (1968*b*) are also shown for comparison.

4. Modified Sachs scaling

An empirical concept which is applied to the calculation of the overpressure for explosions in an exponential atmosphere is that of modified Sachs scaling (see Lutzky & Lehto 1968). According to this concept the overpressure at a distance R from an explosion of energy E in a uniform atmosphere of ambient

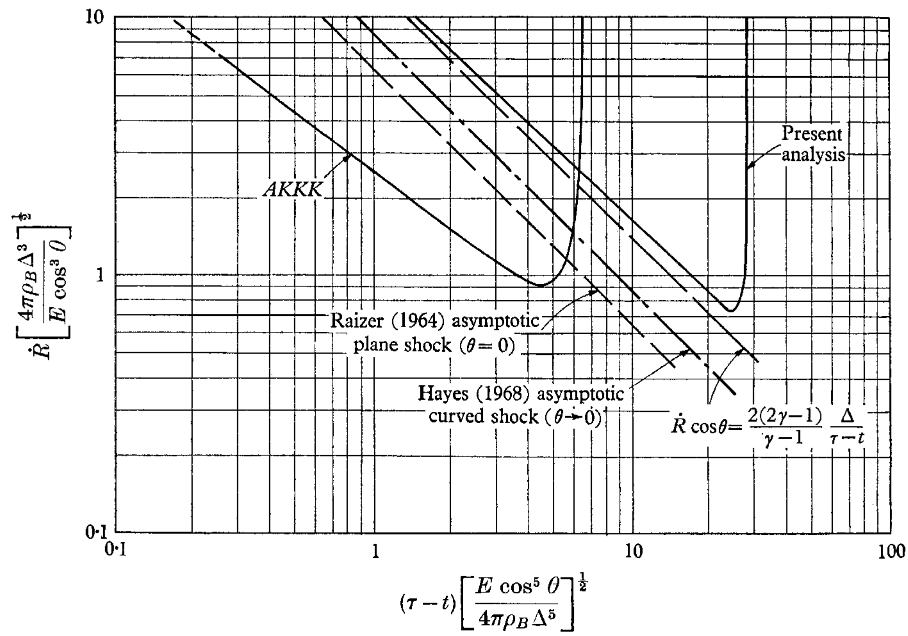

FIGURE 12. Shock velocity in upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$) as a function of time before blowout for $\gamma = 1{\cdot}2$; blowout time $\tau(E\cos^5\theta/4\pi\rho_B\Delta^5)^{\frac{1}{2}} = 28{\cdot}11$.

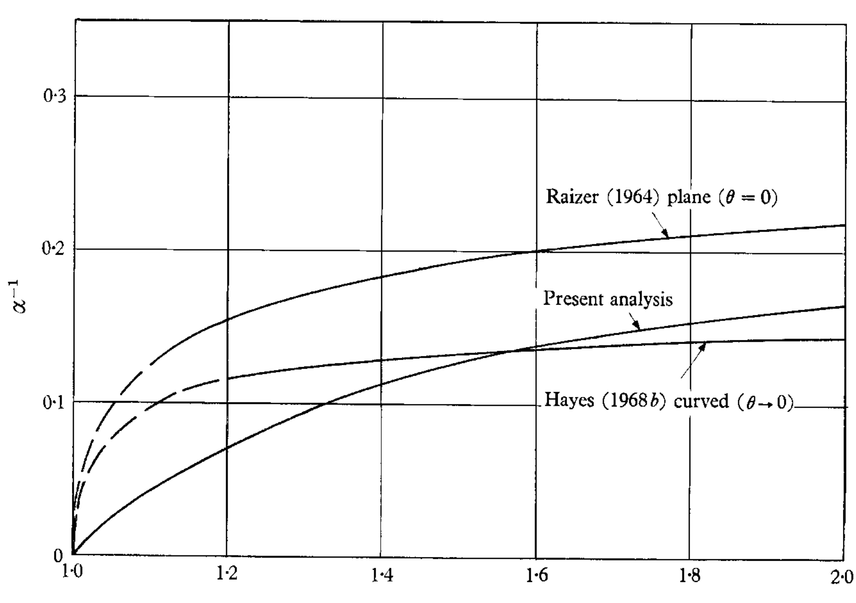

FIGURE 13. Dependence of α on γ in upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$). – – –, extrapolated.

pressure p_0 is the same as the overpressure at a distance R from an explosion of energy E in a non-uniform atmosphere provided that in the non-uniform case the ambient pressure at the distance R is p_0.

Lutzky & Lehto (1967) have noted that modified Sachs scaling agrees quite well with the one-dimensional numerical calculations (Hydrocode) which they have carried out for $\theta > \frac{1}{2}\pi$, with a maximum percentage difference of about 20 %. The reasons for this apparent agreement of modified Sachs scaling with finite difference calculations can be easily understood for the case of a strong shock within the context of the present model.

For a strong shock the overpressure is given by (7), since $p_s \gg p_0$, so that from (26) we have for Δp in the uniform density limit

$$(\Delta p)_{\theta=\frac{1}{2}\pi} \approx p_s = \frac{9E}{4\pi}\frac{(\gamma-1)(\gamma+1)^2}{4\gamma^2-\gamma+3}\frac{1}{R^3}. \tag{43}$$

For η large with $\theta > \frac{1}{2}\pi$ we have for the exponential atmosphere from (16) and (32)

$$(\Delta p)_{\theta>\frac{1}{2}\pi} \approx p_s = \frac{3E}{4\pi}(\gamma-1)(\gamma+1)\frac{1}{R^3}. \tag{44}$$

Numerical results show that in the downward direction p_s lies between the values given by (43) and (44) for all R.

The ratio of the shock pressure given by (43) to that given by (44) is

$$\mathscr{R}_{MS} = \frac{3(\gamma+1)}{4\gamma^2-\gamma+3}. \tag{45}$$

If this ratio were equal to one, then the present model would agree exactly with modified Sachs scaling. This is very nearly the case, since as $\gamma \to 1$ this ratio obviously approaches one. For $\gamma = 1{\cdot}4$, the ratio is 0·765, which no doubt accounts for the maximum variation of 20 % between modified Sachs scaling in the downward direction and the finite difference calculations reported by Lutzky & Lehto. The present analysis shows therefore that for a strong shock modified Sachs scaling gives 'reasonable' results for $\theta \geqslant \frac{1}{2}\pi$, independent of the value of p_0, provided $p_0 \ll \Delta p$.

Comparison of modified Sachs scaling with the present analysis for the upward direction shows a difference of 26 % at $\eta = 2{\cdot}0$ and 38 % at 3·0 for $\gamma = 1{\cdot}4$. The reason for this large and growing deviation with increasing distance can be seen from (16) and (38) which yield

$$(\Delta p)_{\theta<\frac{1}{2}\pi} \approx p_s = \frac{2}{\gamma+1}\left(\frac{\Delta}{\cos\theta}\right)^2 \rho_B k_2 \exp\left(-\frac{\gamma}{2\gamma-1}\eta\right). \tag{46}$$

It is clear that the exponential growth of the shock velocity in the upward direction is not sufficient to offset the exponential decay in the density, with the result that modified Sachs scaling has but a limited range of applicability in the upward direction.

5. Concluding remarks

A model has been developed which predicts the position of the shock envelope as a function of both time and space and which provides an analytic prediction of the flow variables behind the shock. Over the available range of finite difference calculations there is excellent agreement with the present results.

Two scaling laws are presented. The first, which is exact within the theory completely eliminates the parametric dependence on E, Δ and ρ_B and almost eliminates the dependence on θ by transforming the infinity of solutions for various polar angles into two distinct solutions corresponding to the upward and downward motions. The second scaling law, which is semi-empirical, eliminates the primary dependence of the solution on the adiabatic exponent γ over the range of shock positions where the analysis would be expected to apply.

The descending shock solution shows that there is no theoretical limit as to how far downward a strong shock may propagate, contrary to the prediction of Kompaneets (1960) and Andriankin *et al.* (1962). It demonstrates that the upward propagating shock does in fact reach some minimum velocity beyond which it accelerates to infinity in a finite amount of time. It is shown that by relaxing the constant energy constraint for the descending shock, good asymptotic agreement with the exact plane shock solution of Raizer (1963) can be achieved. The far field results for the rising shock are found to agree favourably with the approximate asymptotic curved shock results of Hayes (1968*b*), and with the asymptotic plane shock results of Raizer (1964). The fact that a measure of agreement with the asymptotic calculations is obtained is somewhat surprising in view of the limitations of the model in the far field where it is not expected to apply. The empirical concept of modified Sachs scaling is shown to give reasonable results for $\theta \geqslant \frac{1}{2}\pi$ at all shock locations although for $\theta < \frac{1}{2}\pi$ there is only a limited range of shock positions close to the source where it is applicable.

The integral method which has been used here appears to have promise for the solution of a wide variety of problems in shock dynamics. An important feature of the method is that the approximations to the integrands are inherently exact at the shock front.

The authors would like to express their thanks to Prof. Wallace D. Hayes who read an earlier version of this paper and made many helpful suggestions which contributed greatly to the paper. This work was supported by the Advanced Research Projects Agency (Ballistic Missile Defense Office) and technically administered by the Fluid Dynamics Branch of the U.S. Office of Naval Research under contract Nonr 1841(93). One of the authors (DDL) also acknowledges the support received under a National Aeronautics and Space Administration Traineeship.

Appendix

Eliminating $\partial^2 r/\partial t^2$ between the radial momentum equation (4) evaluated at the shock and (12c) we have

$$-\left.\frac{\partial^2 r}{\partial r_0^2}\right|_R \dot{R}^2 = \frac{2}{\gamma+1}\dot{R}^2\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)_R + \frac{2}{\gamma+1}\ddot{R}, \tag{A 1}$$

where use has been made of the boundary condition (7). On evaluating the pressure-gradient term in the above relation we may insert the resulting value of $(\partial^2 r/\partial r_0^2)_R$ into (12c) to determine the acceleration in terms of R and its derivatives.

From the energy equation (5) and the strong shock condition (6)

$$\left.\frac{1}{p}\frac{\partial p}{\partial r_0}\right|_R = \left(\frac{1}{p_s}\frac{\partial p_s}{\partial r_0} + \frac{\gamma}{\rho}\frac{\partial \rho}{\partial r_0} - \frac{\gamma}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right)_R. \tag{A 2}$$

The first term on the right-hand side may be evaluated from (7) as

$$\left.\frac{1}{p_s}\frac{\partial p_s}{\partial r_0}\right|_R = \left.\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right|_R + \frac{2}{\dot{R}^2}\ddot{R}, \tag{A 3}$$

while from (2)

$$\left.\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right|_R = -\frac{\cos\theta}{\Delta}. \tag{A 4}$$

Finally from the continuity equation (3)

$$\left.\frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\right|_R = \left.\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right|_R + \frac{4}{\gamma+1}\frac{1}{R} - \left.\frac{\gamma+1}{\gamma-1}\frac{\partial^2 r}{\partial r_0^2}\right|_R, \tag{A 5}$$

where $(\partial r/\partial r_0)_R$ has been replaced by (11).

Substituting (A 3) to (A 5) into (A 2) and replacing the pressure gradient term in (A 1) by the resulting expression determines $(\partial^2 r/\partial r_0^2)_R \dot{R}^2$. From (12$c$) the result given by (13c) follows.

REFERENCES

ANDRIANKIN, E. I., KOGAN, A. M., KOMPANEETS, A. S. & KRAINOV, V. P. 1962 The propagation of a strong explosion in a nonhomogeneous atmosphere. *Zh. Prikl. Mekh. Tekh. Fiz.* no. 6, 3–7.

CHERNYI, G. G. 1957 The problem of a point explosion. *Dokl. AN SSSR* **112**, 213–16.

CHERNYI, G. G. 1959 *Introduction to Hypersonic Flow*. English translation (trans. and ed. R. F. Probstein), 1961. New York: Academic Press.

CHERNYI, G. G. 1960 Application of integral relationships in problems of propagation of strong shock waves. *Prikl. Mat. Mekh.* **24**, 121–5; *J. Appl. Math. Mech.* **24**, 159–65.

CHERNYI, G. G. 1961 Integral methods for the calculation of gas flows with strong shock waves. *Prikl. Mat. Mekh.* **25**, 101–7; *J. Appl. Math. Mech.* **25**, 138–47.

GROVER, R. & HARDY, J. W. 1966 The propagation of shocks in exponentially decreasing atmospheres. *Astrophys. J.* **143**, 48–60.

HAYES, W. D. 1968a Self-similar strong shocks in an exponential medium. *J. Fluid Mech.* **32**, 305–15.

HAYES, W. D. 1968b The propagation upward of the shock wave from a strong explosion in the atmosphere. *J. Fluid Mech.* **32**, 317–31.

HAYES, W. D. & PROBSTEIN, R. F. 1966 *Hypersonic Flow Theory*. Vol. I. *Inviscid Flows*, 2nd ed. New York: Academic Press.

KARLIKOV, V. P. 1955 Solution of the linearized axisymmetric problem of a point explosion in a medium with variable density. *Dokl. AN SSSR*, **101**, 1009–12.

KOMPANEETS, A. S. 1960 A point explosion in an inhomogeneous atmosphere. *Dokl. AN SSSR*, **130**, 1001–3; *Soviet Phys. Doklady*, **5**, 46–8.

LUTZKY, M. & LEHTO, D. L. 1968 Shock propagation in spherically symmetric exponential atmospheres. *Phys. Fluids*, **11**, 1466–1472.

RAIZER, YU. P. 1963 Motion produced in an inhomogeneous atmosphere by a plane shock of short duration. *Dokl. AN SSSR*, **153**, 551–4; *Soviet Phys. Doklady*, **8**, 1056–8 (1964).

RAIZER, YU. P. 1964 The propagation of a shock wave in a nonhomogeneous atmosphere in the direction of decreasing density. *Zh. Prikl. Mekh. Tekh. Fiz.* no. 4, 49–56.

SEDOV, L. I. 1946 Propagation of strong blast waves. *Prikl. Mat. Mekh.* **10**, 241–50.

SEDOV, L. I. 1957 *Similarity and Dimensional Methods in Mechanics*, 4th edition, English transl. (ed. M. Holt), 1959. New York: Academic Press.

TAYLOR, G. I. 1950 The formation of a blast wave by a very intense explosion. *Proc. R. Soc.* A**201**, 159–86.

TROUTMAN, W. W. & DAVIS, C. W. 1965 The two-dimensional behavior of shocks in the atmosphere. *Air Force Weapons Lab., Rept.* AFWL-TR-65-151.

ZEL'DOVICH, YA. B. & RAIZER, YU. P. 1966 *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, 2nd ed. English translation (ed. W. D. Hayes and R. F. Probstein), vol. II, 1967. New York: Academic Press.